Imperial College London

Lecture 5

Time-domain analysis: Convolution

(Lathi 2.4)

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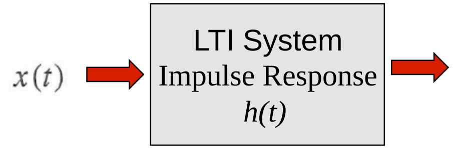
Convolution Integral

- **Convolution Integral:**

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

- System output (i.e. zero-state response) is found by convolving input x(t) with System's impulse response h(t).



$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

Convolution Table (1)

- Use table to find convolution results easily:

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$	
1	$x(t)$	$\delta(t-T)$	$x(t-T)$	
2	$e^{\lambda t}u(t)$	$u(t)$	$\frac{1-e^{\lambda t}}{-\lambda}u(t)$	
3	$u(t)$	$u(t)$	$tu(t)$	
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_1-\lambda_2}u(t)$	$\lambda_1 \neq \lambda_2$
5	$e^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$te^{\lambda t}u(t)$	
6	$te^{\lambda t}u(t)$	$e^{\lambda t}u(t)$	$\frac{1}{2}t^2e^{\lambda t}u(t)$	

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Convolution Table (2)

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
7	$t^N u(t)$	$e^{\lambda t}u(t)$	$\frac{N!e^{\lambda t}}{\lambda^{N+1}}u(t) - \sum_{k=0}^{N}\frac{N!t^{N-k}}{\lambda^{k+1}(N-k)!}u(t)$
8	$t^M u(t)$	$t^N u(t)$	$\frac{M!N!}{(M+N+1)!}t^{M+N+1}u(t)$
9	$te^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_2 t}-e^{\lambda_1 t}+(\lambda_1-\lambda_2)te^{\lambda_1 t}}{(\lambda_1-\lambda_2)^2}u(t)$
10	$t^M e^{\lambda t}u(t)$	$t^N e^{\lambda t}u(t)$	$\frac{M!N!}{(N+M+1)!}t^{M+N+1}e^{\lambda t}u(t)$
11	$t^M e^{\lambda_1 t}u(t)$	$t^N e^{\lambda_2 t}u(t)$	$\sum_{k=0}^{M}\frac{(-1)^k M!(N+k)!t^{M-k}e^{\lambda_1 t}}{k!(M-k)!(\lambda_1-\lambda_2)^{N+k+1}}u(t)$
	$\lambda_1 \neq \lambda_2$		$+\sum_{k=0}^{N}\frac{(-1)^k N!(M+k)!t^{N-k}e^{\lambda_2 t}}{k!(N-k)!(\lambda_2-\lambda_1)^{M+k+1}}u(t)$

Convolution Table (3)

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
12	$e^{-\alpha t}\cos(\beta t+\theta)u(t)$	$e^{\lambda t}u(t)$	$\frac{\cos(\theta-\phi)e^{\lambda t} - e^{-\alpha t}\cos(\beta t+\theta-\phi)}{\sqrt{(\alpha+\lambda)^2+\beta^2}}u(t)$ $\phi = \tan^{-1}[-\beta/(\alpha+\lambda)]$
13	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(-t)$	$\frac{e^{\lambda_1 t}u(t)+e^{\lambda_2 t}u(-t)}{\lambda_2-\lambda_1}$ $\quad \text{Re}\,\lambda_2 > \text{Re}\,\lambda_1$
14	$e^{\lambda_1 t}u(-t)$	$e^{\lambda_2 t}u(-t)$	$\frac{e^{\lambda_1 t}-e^{\lambda_2 t}}{\lambda_2-\lambda_1}u(-t)$

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Example (1)

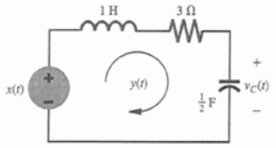
Find the loop current y(t) of the RLC circuits for input $x(t) = 10e^{-3t}u(t)$ when all the initial conditions are zero.

- We have seen in slide 4.5 that the system equation is:

$$(D^2 + 3D + 2)y(t) = Dx(t)$$

- The impulse response h(t) was obtained in 4.6:

$$h(t) = (2e^{-2t} - e^{-t})u(t)$$



- The input is: $x(t) = 10e^{-3t}u(t)$
- Therefore the response is:

$$\begin{aligned} y(t) &= x(t) * h(t) \\ &= 10e^{-3t}u(t) * [2e^{-2t} - e^{-t}]u(t) \end{aligned}$$

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Example (2)

- Using distributive property of convolution:

$$\begin{aligned} y(t) &= 10e^{-3t}u(t) * 2e^{-2t}u(t) - 10e^{-3t}u(t) * e^{-t}u(t) \\ &= 20[e^{-3t}u(t) * e^{-2t}u(t)] - 10[e^{-3t}u(t) * e^{-t}u(t)] \end{aligned}$$

- Use convolution table pair #4:

$$\begin{aligned} y(t) &= \frac{20}{-3-(-2)}[e^{-3t} - e^{-2t}]u(t) - \frac{10}{-3-(-1)}[e^{-3t} - e^{-t}]u(t) \\ &= -20(e^{-3t} - e^{-2t})u(t) + 5(e^{-3t} - e^{-t})u(t) \\ &= (-5e^{-t} + 20e^{-2t} - 15e^{-3t})u(t) \end{aligned}$$

No.	$x_1(t)$	$x_2(t)$	$x_1(t) * x_2(t) = x_2(t) * x_1(t)$
4	$e^{\lambda_1 t}u(t)$	$e^{\lambda_2 t}u(t)$	$\frac{e^{\lambda_1 t} - e^{\lambda_2 t}}{\lambda_1 - \lambda_2}u(t)$ $\quad \lambda_1 \neq \lambda_2$

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When input is complex ……

- What happens if input $x(t)$ is not real, but is complex?
- If $x(t) = x_r(t) + jx_i(t)$, where $x_r(t)$ and $x_i(t)$ are the real and imaginary part of $x(t)$, then

$$\begin{aligned} y(t) &= h(t) * [x_r(t) + jx_i(t)] \\ &= h(t) * x_r(t) + jh(t) * x_i(t) \\ &= y_r(t) + jy_i(t) \end{aligned}$$

- That is, we can consider the convolution on the real and imaginary components separately.

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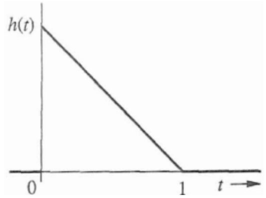
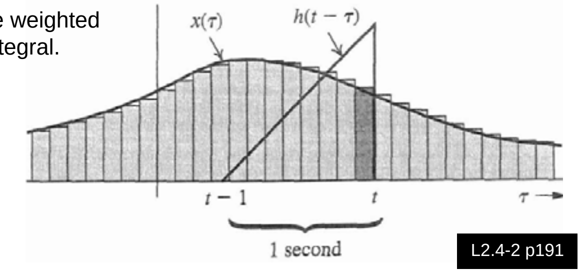
Intuitive explanation of convolution

- Assume the impulse response decays linearly from t=0 to zero at t=1.
- Divide input x(τ) into pulses.
- The system response at t is then determined by x(τ) weighted by h(t- τ) (i.e. x(τ) h(t- τ)) for the shaded pulse, PLUS the contribution from all the previous pulses of x(τ).
- The summation of all these weighted inputs is the convolution integral.





$$y(t) = x(t) * h(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

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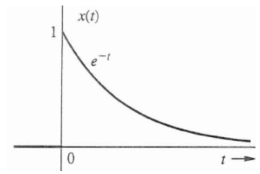
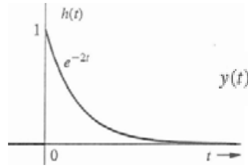
Convolution using graphical method (1)

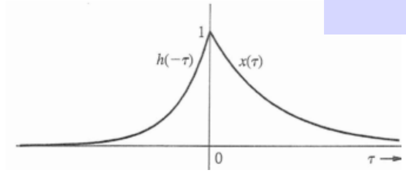
Determine graphically $y(t) = x(t)*h(t)$ for $x(t) = e^{-t}u(t)$ and $h(t) = e^{-2t}u(t)$.





$$y(t) = \int_0^t x(\tau)h(t-\tau)\,d\tau \qquad t \geq 0$$

Remember: variable of integration is τ, not t



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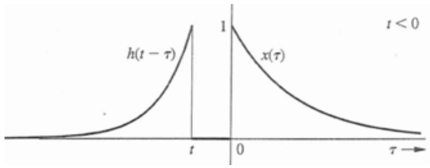
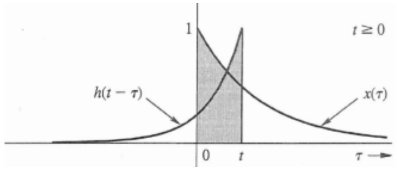
Convolution using graphical method (2)

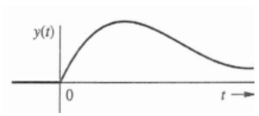
$$y(t) = \int_0^t e^{-\tau} e^{-2(t-\tau)}\,d\tau$$
$$= e^{-2t}\int_0^t e^{\tau}\,d\tau$$
$$= e^{-t} - e^{-2t}$$





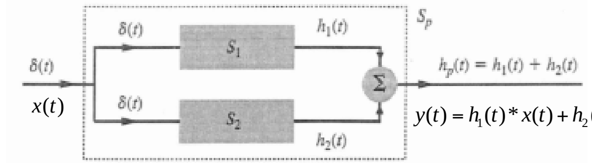
Moreover, $y(t) = 0$ for $t < 0$, so that

$$y(t) = (e^{-t} - e^{-2t})u(t)$$



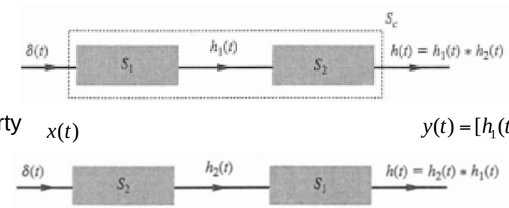
Interconnected Systems

- Parallel connected system



$$y(t) = h_1(t) * x(t) + h_2(t) * x(t)$$

- Cascade systems &
- Commutative property



$x(t)$

$$y(t) = [h_1(t) * h_2(t)] * x(t)$$

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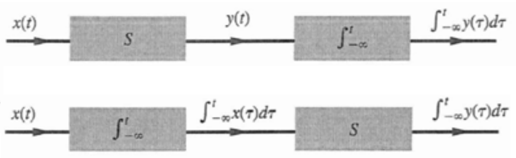
Interconnected Systems

- Integration:

  if $\quad x(t) \Longrightarrow y(t)$

  then $\quad \int_{-\infty}^{t} x(\tau)\,d\tau \Longrightarrow \int_{-\infty}^{t} y(\tau)\,d\tau$



- Also true for differentiation:

  if $\quad x(t) \Longrightarrow y(t)$

  then $\quad \frac{dx(t)}{dt} \Longrightarrow \frac{dy(t)}{dt}$

- Let $x(t) = \delta(t)$ and $y(t) = h(t)$ (x(t) is an impulse, and h(t) is the impulse response of the system)
- Then $g(t)$, the step response is: $g(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$

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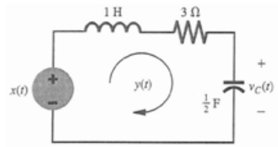
Total Response

Total response = zero-input response + zero-state response

$$\text{total response} = \underbrace{\sum_{k=1}^{N} c_k e^{\lambda_k t}}_{\text{zero-input component}} + \underbrace{x(t) * h(t)}_{\text{zero-state component}}$$

- Let us put everything together, using our RLC circuit as an example.
- Let us assume $x(t) = 10e^{-3t}u(t), y(0) = 0, \quad \dot{y}(0) = -5.$
- In earlier slides, we have shown that



$$\text{total current} = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input current}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state current}} \qquad t \geq 0$$

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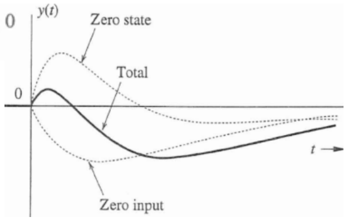
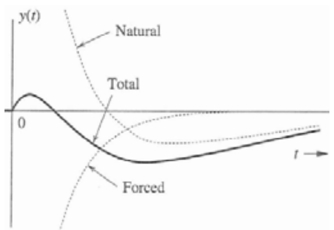
Natural vs Forced Responses

$$\text{total current} = \underbrace{(-5e^{-t} + 5e^{-2t})}_{\text{zero-input current}} + \underbrace{(-5e^{-t} + 20e^{-2t} - 15e^{-3t})}_{\text{zero-state current}} \qquad t \geq 0$$



- Note that characteristic modes also appears in zero-state response (because it has an impact on h(t)).
- We can collect the e^{-t} and e^{-2t} terms together, and call these the **NATURAL response**.
- The remaining e^{-3t} which is NOT a characteristic mode is the **FORCED response**.



$$\text{total current} = \underbrace{(-10e^{-t} + 25e^{-2t})}_{\text{natural response } y_n(t)} + \underbrace{(-15e^{-3t})}_{\text{forced response } y_\phi(t)} \qquad t \geq 0$$

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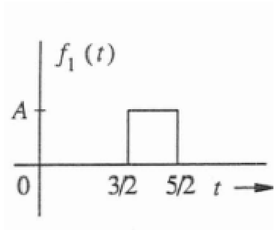
Additional Example



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